5. Simple Interest and Compound Interest

Ramus ... boasted of his familiarity with all the shops and market-places in Paris, where mathematics was put to use. He held that only those parts of current mathematics were worth teaching that had a demonstrable practical application. Mathematics as studied in universities should not be more than a systematic treatment of mathematical methods in use by merchants, navigators, surveyors and engineers. (van Berkel 1988, p.157)

Basic Interest Concepts

Interest is fundamental to the economic process. The production of goods and services requires the combination of capital and labour, with ('originary') interest arising from the return to invested capital. When capital is securitized into financial instruments, such as annuities, bills of exchange, or joint stocks, ('loan') interest is associated with compensating the owner of the capital for foregoing the return obtainable elsewhere (Böhm-Bawerk 1914, p.6). The presence of interest necessitates the development of techniques for calculating interest. One type of interest calculation concerns the division of profits from a business venture. For example, if a group of merchants form a partnership in which capital is invested for different periods of time, some method is required for determining the share (interest) of each partner when the partnership is dissolved.

Various types of interest calculations arise in the analysis of financial securities. A simple and important historical example involved the use of payback or '*years' purchase*' (= Price/annuity payment) to reflect relative annuity value. Years' purchase was also used to value other types of fixed income cash flows. Years' purchase was used as a measure of value whether the annuity was redeemable, perpetual, or for life. For a perpetual annuity, years' purchase would be the inverse of the true yield interest rate. However, for other types of annuities the precise relation between years' purchase and the offered interest rate is not as clear. Yet, on balance, years' purchase does represent a quick and useful method of determining relative value, though the practice of using years' purchase is only a crude method of arriving at an interest rate measure. A form of years' purchase, the current yield, which can be used to assess relative bond values, survived until the advent of modern computers permitted ready calculation of the yield to maturity.

The Relationship Between Years' Purchase and Yield to Maturity

Years' purchase differs from yield to maturity in many cases. These differences are readily identified by first considering two cases where the measures are equal: for perpetual annuities; and for securities selling at par. The case of securities selling at par is applicable to instruments such as bonds, which offer a fixed term annuity of coupon payments (C) with a return of principal (M) at maturity. If the price of the bond is par, then the yield to maturity (y) will equal C/M. But, in the par bond case, $P/C = M/C = 1/y$. Similarly, a perpetual annuity payment of A, paid each year in perpetuity, has a price of $P = A/y$. Again, $P/A = 1/y$ and years' purchase and yield to maturity provide the same measure of value.

It follows that years' purchase and yield to maturity differ for annuities with fixed terms and securities selling off-par. This difference will be greatest when the term to maturity is small and the difference between par value and price is large. In the context of the current yield, an extension of the years' purchase measure, Francis (1983, p.27) provides a graphical presentation of the relevant functional relationship.

Various methods can be used to compare the relative values of fixed income cash flows. In the less developed capital markets of the 16th to 18th centuries, the use of years' purchase avoided the confusion associated with quoting different valuations for the same security. The stated interest rate on a fixed income security determined the annuity payment, which invariably involved multiplying the principal value by the stated interest rate to determine the payment. However, if the annuity was sold at a price other than the par value, the stated interest would differ from the yield to maturity, another interest rate measure. Another related example arises with the calculation of loan interest. If the stated interest rate on a loan of T years is 10%, the size of the regular loan payment would differ depending on whether simple or compound interest is used, as well as the compounding frequency if compound interest is involved.

Like other commodities, financial securities are traded in the market on the basis of price. In many situations, the price does not provide a ready measure for comparing the relative values of different securities. In modern financial markets, interest calculations are used to provide a more useful measure of value. One example where interest rate comparisons are useful occurs with financial securities that offer cash flows differing in size and timing, such as an annuity certain paying $\$A$ for T years and a zero coupon bond paying $\$Y$ in T years.[1] Comparing quoted security prices does not provide a useful method of comparing value in these cases. Using the quoted prices, the interest rates offered by each security can be determined and used to compare the promised cash flows. A variety of different methodologies for determining the 'interest rate' are available.

While usually guided by common sense, the interest calculation method used in a specific situation depends upon convention. For example, in the US, the modern market convention is to quote discount rates for trading money market securities. Discount rates are not directly comparable to true yields, for example, Stigum (1982).[2] While a true yield calculation would appear to be more theoretically sensible, the convention of using discount rates has a long history, dating at least from the early trading of bills of exchange in the Middle Ages. Other examples of unusual conventions abound. In Canada, the legally defined convention for quoting mortgage interest rates is to use semi-annual compounding, even though payments are made monthly. Also, in the various instances where simple interest calculations are the convention, compounding is ignored and 'profit on profit' is not recognized.

An important topic in the early history of financial economics is the evolution of particular methods of determining interest. Conventions such as the use of years' purchase provided guidance to market participants, but years' purchase lacked precision and was not applicable to all security cash flows, for example, zero coupon instruments. The evolution of interest calculations involved increasing mathematical sophistication. Acceptance into market practice required a general level of sophistication on the part of market participants. In this process, the reckoning masters and algorists played a key role. The commercial arithmetics reflected the sophistication that had been achieved at a given point in time. From these sources, it would appear as though the rule of three was in common use for interest calculations at the end of the 15th century. The use of compound interest calculations was becoming widespread and well developed by the end of the 16th century.

Methods of Fixed Income Valuation

Allowing for deviations due to convention, the modern valuation of fixed income cash flows involves both simple and compound interest.[3] The value or price is calculated for a specific point in time. A present value calculation provides the current price (present value) and a future value calculation provides the price at some future time (future value). Present value is determined by discounting future cash flows, which involves dividing each future cash flow by the compound interest factor appropriate for that time period. Future value is determined by multiplying (compounding) each future cash flow by the compound interest factor appropriate for the future time period where a price is desired. Consistency requires that, for each specific time period, the compound interest factor used for present and future value calculations be directly related. Hence, future value and present value are inverse operations. Because the current price is usually desired, present value calculations are most commonly done.

From the generic present value and future value concepts, four basic types of valuation problems can be distinguished: present value of single cash flows; future value of single cash flows; present value of a sequence of equal cash flows (annuity payments) that are paid at equal intervals for T periods, usually referred to as annuities or annuities certain; and, future value of a sequence of equal cash flows (annuity payments). Valuation requires knowledge of: the price; the size of the payment; the time period (term to maturity); and the interest (discount) rate. Theoretically, provided with information on any three variables it is possible to solve for the fourth variable. In the easiest problems to solve, the term to maturity and interest rate are given and the problem is to solve for the price given the size of the annuity payment (a present value problem) or to determine the annuity payment required to achieve a given terminal value (a future value problem). Because present and future value are inverse problems, the same general methodology is used to solve either problem.

The evolution of financial economics can be assessed relative to which valuation techniques were in common practice at a particular time. Many early calculations involved the use of simple interest, where the compounding of interest, the payment of interest on

What is the Rule of Three?

The rule of three is basic to problems involved in exchange. The rule was considered so important in commercial arithmetic that it was often referred to as 'the golden rule' or 'the merchant's rule'. The rule involves the different variations on proportions: a is to b as c is to x, using a, b and c to solve for x:

$$(a/b) = (c/x) \quad \rightarrow \quad [(b)(c)]/a = x$$

The *Treviso Arithmetic* gives the following example: 'If 1 pound of saffron is worth 7 pounds of pizoli, what will 25 pounds of the same saffron be worth?' Observing that $a = 1$, $b = 7$, and $c = 25$, the solution $x = 175$ pounds of pizoli (not stated in the *Treviso*) follows from application of the rule. The *Treviso* goes on to clear up any possible confusion about the terminology: 'So this is not called the rule of the three things because there are three things of different nature, but because one thing is mentioned twice.'

The different possible variations on the rule of three were typically taught in rote fashion, without reference to the basic algebraic foundation. The application of the Rule of Three to situations such as currency exchange, barter, and calculating of price is apparent. For example, the *Treviso* gives the following problem: 'If 100 pounds of sugar are worth 32 ducats, what will 9812 pounds be worth?' In this case, application of the rule of three produces the solution 3139 ducats and 84 remainder which must be further manipulated to express the 84 remainder in terms of grossi and pizoli. The method of doing so is also illustrated and the final solution of 3139 ducats, 20 grossi, 5 pizoli and remainder is determined.

The origins of the Rule of Three can be traced at least to 1650 BC, though the reference to the 'three rule' appears to have be derived from the seventh century Hindu mathematician Brahmagupta (Swetz 1987, p.225). Over time, the Rule of Three was applied to areas other than strictly mercantile calculations. Klein (1996, ch. 2) examines 17th and 18th century extensions of the Rule of Three to subjects such as political arithmetic, demographics and other areas.

accumulated interest, is ignored. Simple interest is still conventional practice in modern financial markets for determining prices of fixed income securities with less than one year to maturity. With simple interest, the payout at maturity is determined by multiplying the principal value by one plus the stated interest rate (expressed as a decimal fraction). Where an annualized interest rate is stated and the time period of the investment is less than one year, the interest rate is adjusted to reflect the length of time the investment is outstanding. For example, if the annualized simple interest rate is i and there are n months to the maturity date, when a single cash flow of $\$A$ will be paid, then the present value (P) or price of the security, at simple interest, is:

$$P = \frac{\$A}{1 + i\,\frac{n}{12}}$$

Simple interest is theoretically appealing for valuing single cash flows that are to be received within a year or less.

Simple interest can lead to complications when the time period for the investment is greater than a year because of the method used to account for the payment of compound interest or 'profit on profit'. In the case of simple interest, the convention is to make no compounding payments. For example, if the *single* cash flow of $\$A$ of the previous example is to be received in T years, instead of n months, then the payment of $\$A$ at maturity would determine the price because $\$A$ would have to equal $T(iP) + P$, or:

$$P = \frac{\$A}{1 + T\,i}$$

This present value represents the accumulation of interest credits (iP) over T years and the return of initial investment of P.

Per se, ignoring the payment of 'profit on profit' does not imply that simple interest is theoretically flawed. In many situations, the method of calculating interest has more to do with convention than with actual impact on the returns from financial contracts. The interest convention used determines the method for calculating the current price associated with the final payout stated in a financial contract. For the example of $\$A$ to be paid in T years, both compound and simple interest can produce the same present value (P) by adjusting the stated interest rate. More generally, as long as the interest rate is calculated from an observed price, even when the maturity date and final payout are not known when the financial contract is initiated, the parties to the contract can still make informed decisions whatever the method used for calculating returns.

Despite this, there are situations where important discrepancies can arise between compound and simple interest calculations. In particular, consider the problem of calculating the value of shares in a partnership where the partners have participated for different lengths of time. By ignoring 'profit on profit', simple interest will undervalue the shares of those partners who have had funds invested for the longest period of time. Similarly, simple interest would tend to favour partners who took funds out of the enterprise over time. Attempts to account for the discrepancy between simple and compound interest solutions by adjusting the relative value of partners' shares would be difficult, at best. The commercial incentives to use compound interest in calculating the value of shares in partnerships are considerable.

One important advantage of simple interest is that calculations can be done using the basic arithmetical operations of multiplication and division. While more theoretically satisfying, compound interest does involve the evaluation of powers and roots. However, with the use of compound interest tables, it is possible to transform standardized compound interest problems into a form that can be handled using only multiplication and division. To see this consider the basic problem of determining the price to be paid today for the receipt of $\$M$ in T years at a compound annual interest rate of $r\%$:

$$P = \frac{\$M}{(1 + r)^T} = (\$M)\left[\frac{1}{(1 + r)^T}\right] \equiv (\$M)\ (PVIF \mid T, r)$$

The preparation of early tables was facilitated by market conventions and legal restrictions that required the use of only a small number of possible interest rates.

Widespread adoption of compound interest for simple cash flows required the use of tables that contained the PVIF (present value interest factor for a payment of $1) numbers needed to simplify the calculations for selected $\{T, r\}$ combinations. Inversion of the PVIF number gives the associated FVIF (future value interest factor for a payment of $1):

$$\frac{P}{(PVIF \mid T, r)} = \$M = P\ (1 + r)^T \quad \rightarrow$$

$$(1 + r)^T = (FVIF \mid T, r) = (PVIF \mid T, r)^{-1}$$

With some additional effort, PVIF tables could be used to evaluate more complicated compound interest problems, such as pricing of annuities. This could be done by brute force, discounting each individual cash flow. However, in cases where annuity cash flows are equal until the term to maturity, a simplification derived by applying the solution to a geometric progression is available. This simplification is not obvious and, judging from the solution to problems presented by Jean Trenchant, did not appear to be well understood around the mid-16th century. Yet, the result was known and correct solutions using PVIFA (present value interest factor for an annuity of $1) are presented in other 16th century works such those by Simon Stevin.

To see the simplification required, consider the problem of determining the price of a fixed term annuity, a stream of equal payments of $\$A$ made each year for T years at $r\%$:

$$\textit{Annuity Price} \equiv (\$A)\ [PVIFA \mid T, r]$$

$$= \frac{\$A}{(1+r)} + \frac{\$A}{(1+r)^2} + \frac{\$A}{(1+r)^3} + \ldots + \frac{\$A}{(1+r)^T}$$

$$= \$A \sum_{t=1}^{T} \frac{1}{(1+r)^t} = (\$A)\left[\frac{1}{r} - \frac{1}{r\,(1+r)^T}\right]$$

Similarly, the FVIFA can also be calculated. The FVIFA is used to determine the future value of series of equal payments, paid into a fund, made each year for T years with the first payment starting in year 1 and continuing each year until the final payment is made on the term to maturity date:

$$\textit{Future Value of Annuity} \equiv (\$A)\ [FVIFA \mid T, r]$$

$$= \$A\ (1+r)^{T-1} + \$A\ (1+r)^{T-2} + \ldots + \$A\ (1+r) + \$A$$

$$= \$A \sum_{t=0}^{T-1} (1+r)^t = (\$A)\ \{\frac{(1+r)^T - 1}{r}\}$$

From this the relation between PVIFA and FVIFA follows:

$$(PVIFA \mid T, r) = \frac{1}{r}\ [1 - \frac{1}{(1+r)^T}] = \frac{(FVIFA \mid T, r)}{(1+r)^T}$$

It follows that PVIF and PVIFA tables can be used to construct FVIF and FVIFA values so only one pair of tables is required.

Another useful mathematical manipulation involves applying the closed form for a geometric series $[1/(1 - x)]$ to solve the pricing formula for a perpetual annuity:[4]

$$\sum_{t=1}^{\infty} \frac{1}{(1+r)^t} = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \dots.$$

$$= \frac{1}{1+r}\left\{1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots\right\}$$

$$= \frac{1}{1+r}\left\{\frac{1}{1 - \frac{1}{1+r}}\right\} = \frac{1}{r}$$

Observing that $0 < r < 1$ which is sufficient to ensure convergence of the geometric series, the pricing formula for a perpetual annuity follows immediately from applying the solution for a geometric series to the perpetual cash flow of \$1. Recognizing the solution for the perpetual, the PVIFA factor could be derived without much effort using the closed form for a geometric progression of T terms: $(1 - x^T)/(1 - x)$. Using this approach is instructive because it permits PVIFA values to be calculated, without substantial effort, if only PVIF tables are available. However, if numerous PVIFA calculations are needed, tables containing the PVIFA numbers do substantially simplify the requisite calculations.

Origins of Some Mathematical Tools

An important topic in the early history of financial economics is the process underlying the transition from simple to compound interest calculations. There is historical evidence indicating that compound interest was used in business transactions going back to ancient times, for example, Divine (1959). However, over time there was considerable social resistance to compound interest, as reflected in various legal and ecclesiastic prohibitions. At the beginning of the 16th century, methods for calculating interest taught by reckoning masters typically involved simple interest and the rule of three. Where compound interest was used in Christian business, the practice was not advertised, though there is evidence that groups not subject to canon law, such as Jewish money lenders, did use compound interest.

The transition of market practice from simple to compound interest raises a number of interesting questions for study. One such question concerns the impact of social restrictions, such as the scholastic usury doctrine, on the recorded methods for calculating interest. To what extent were social restrictions responsible for the use of simple interest, as reflected in the commercial arithmetics? Was the proposed use of simple arithmetic a ruse, with compound interest being the conventional but unstated commercial practice? Or, were reckoning masters, bound by training using operations such as the rule of three, unable to handle the more advanced calculations required to determine compound interest? Another related question concerns the mathematical prerequisites needed to do compound interest calculations. Did reckoning masters have the more advanced mathematical training to undertake such alternative calculations?

Resolving questions surrounding the transition from simple to compound interest calculations is not easy. There is no specific landmark text that can be pointed to as a benchmark for the change in methods. The potential civil and ecclesiastical penalties associated with violating Church doctrine on usury meant that merchants disguised interest payments in various types of seemingly nonusurous contracts, such as bills of exchange. However, it is possible to make some indirect inferences based on information about the state of mathematical knowledge. If the typical reckoning master was not equipped to understand compound interest calculations, then it is unlikely that compound interest would be a method that was widely used to determine a 'fair and just' return on a fixed income investment or partnership.

Significantly, the use of compound interest can require more than the mechanical ability to do compound interest calculations. Mechanical use of compound interest requires no more than the simple ability to do operations such as multiplications using PVIF and PVIFA factors. However, understanding the logic of compound interest operations involves knowledge of powers, roots and some elements of algebra. Widespread use of compound interest calculations would also mean that, at least, PVIF tables would be found in historical manuscripts. Because the availability of compound interest tables is almost essential to the widespread implementation of compound interest calculations, the availability and sophistication of tables for compound interest calculations can be used as a partial measure of the extent to which compound interest was used in general business practice.

As for the mathematical knowledge needed, the arithmetic required for simple interest calculations, such as the rule of three, involve mathematics that were available to ancient civilizations. Evolution of this aspect of commercial arithmetic is more associated with the progressive adoption of Hindu-Arabic numerals and algorithms replacing Roman numerals and the abacus. The mathematics needed for basic compound interest calculations were also available from ancient times. Calculation of powers such as $(1 + r)^n$ is an extension of multiplication and the process of discounting involves only division. The convention of using only a limited range of (percentage) integer valued interest rates meant that a reckoning master, if need be, could prepare rudimentary tables for selected compound interest calculations without substantial effort.

Reckoning masters almost certainly had the mathematical knowledge and ability to handle a restricted range of compound interest calculations. However, there were many types of compound interest problems that would pose considerable difficulty. For example, the problem of solving for an interest rate, given the initial investment, cash flows, and term to maturity, could be handled by trial and error if the cash flows were small in number. More complicated cash flow situations could be handled if sufficiently detailed tables were available. Because the general problem of solving for a specific interest rate by trial and error requires tables covering a wide range of $\{T, r\}$ combinations, such tables would have to be more detailed than would be practical for a reckoning master to construct on their own. As noted previously, the availability and sophistication of such tables is a reflection of the acceptance and use of compound interest calculations.

The decidedly more complicated mathematical problem of solving for an interest rate algebraically, as the root of an algebraic equation, was restricted to only the simplest type of problems, if it was used at all. Adequate algebraic methods for general solutions of the roots of cubic and quartic equations did not appear until the 16th century with the contributions of Tartaglia, Cardano and Ferrari, though less general methods suitable for compound interest problems were available much earlier. Up to modern times, there is little evidence that direct algebraic methods ever had any popularity as a method of solving for a specific interest rate in compound interest problems, except as mathematical exercises. Until the advent of modern computers, trial and error, using tables combined with interpolation, was the preferred approach for almost all practical applications.

The widespread use of annuities in lending and borrowing raises a question about the use of compound interest in calculating prices for this type of cash flow pattern. The mathematics required to reduce the geometric progression associated with an annuity to the formula $[(1/r) - \{1/(r(1+r)^T)\}]$ were known at least since Euclid (365?-?), and perhaps as early as Babylonian times. Proposition 35 of Book IX of Euclid's *Elements* states the following (Boyer 1968, p.127) abstract solution to the sum of a geometric progression: 'If as many numbers as we please be in continued proportion, and there be subtracted from the second and last numbers equal to the first, then as the excess of the second is to the first, so will the excess of the last be to all those before it.' With some manipulation, the stated solution to geometric progression problem can be applied to derive the familiar annuity formula.

Euclid's Book IX, Proposition 35 rule was certainly known to medieval writers (D. Smith 1958, v.2, p.502). The rule appears in the *Liber abaci* of Fibonacci with more modern treatments appearing in 15th century Italian algebras. Chuquet states the rule in the *Triparty* in a form that is readily adaptable to annuity calculations. Various other presentations appear in the 16th century, by which time problems involving the geometric progression were popular in mathematical texts. The mathematics for the infinite form of the geometric progression, the geometric series, was first examined by Archimedes around 225 B.C. with the general formula being given by Vieta (Francois Viete, 1540-1603). The similarity of the price of a perpetual annuity and the price of an annuity with a very distant term to maturity makes it likely that, given a specific interest rate and term to maturity, reckoning masters were able to price perpetuities and annuities using compound interest methods.

Some forms of compound interest calculations are solvable using logarithms. Immediate examples of problems where logarithms would ease the computational burden are: solving for the compound interest rate of a single cash flow occurring in T periods; and, solving compound interest rate problems involving less than one year to maturity. The Scot, John Napier (1550-1617), is credited with discovering the logarithm, with the discovery first appearing in his *Descriptio* (1614) after twenty years of work. Napier developed logarithms to primarily solve problems involving multiplication in trigonometry but it was not long before individuals such as Henry Briggs and Edward Wright extended the concept to base 10 and base e.

Halley (1761) on the formula for Valuing the Annuity

'Of Compound Interest' is a scarce contribution by Edmond Halley which appeared in *Sherwin's Mathematical Tables* (1761). The paper deals primarily with the use of logarithms to solve various present value problems. In the analysis of annuity pricing, Halley describes the derivation of the formula for the annuity. As is typical of works from this time, 'r' is used to denote $(1 + i)$, where i is the rate of interest:

> Let a be an annuity or yearly pension whose successive amounts for times past are ar^t, and whose present values are a/r^t successively, by what goes before; and the series, &c., ar^5, ar^4, ar^3, ar^2, ar, a, a/r, a/r^2, a/r^3, a/r^4, a/r^5, &c., will be a rank of means proportionals continued infinitely in the ratio of r to 1. Now the sum of all the consequents, or of the whole infinite series, will be to the said sum increased by the next greater term (or the sum of all the antecedents) as to 1 to r, by *Euclid*, v.12; wherefore, putting y for the sum of said consequents, ry will be equal to $y + ar^t$, the sum of the antecedents; and $ry - y = ar^t$; and, therefore, $ar^t/(r - 1)$ will be equal to y, the sum of all our mean proportionals whereof ar^{t-1} is the greatest; and by the same rule, $a/(r - 1)$ will be the sum of all the terms whereof a/r is the greatest. So that if we abstract $a/(r - 1)$ from $ar^t/(r-1)$, the difference will be the sum of all the terms whereof ar^{t-1} is the greatest and a the least, their number being t, which sum we will call z; therefore z (the amount of the annuity of a foreborne t times at the rate r) $= (ar^t - a)/(r - 1)$.

Halley's statement of the now familiar formula for the annuity reflects a form and manner which is similar to the spirit of the original Euclid. Why was this derivation included by Halley? Presumably, because he felt that the result was not familiar enough in his time to be taken as given. This feeling was not confined to Halley. Price (1772), for example, included a similar discussion in his Appendix which contains tables for present value and future value calculations.

By the middle of the century, the logarithm was well established in Europe and applications of logarithms were appearing in published arithmetics.

Though not of central importance to the mathematics used in interest calculations, the genesis of the series solution to log[1 + x] is interesting.[5] It is conventional in modern texts to derive this result as an elementary application of Taylor's theorem, which was initially presented by Brook Taylor (1685-1731) in 1715. However, the series approximation:

$$\log[1 + x] = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

was presented much earlier by the Danish mathematician Nicolaus Mercator (1620-1687) in *Logarithmotechnia* (1668).[6] While this infinite series solution does appear to differ from the Taylor series solution where the denominators in the nth series term is n!, it is possible to show the two series are equivalent. The form of the series solution given above is conventionally referred to as a Mercator series (D. Smith 1958, v.2).

At what point did the mathematical content of commercial arithmetic cease to progress? On this point, D. Smith (1958, v.1, p.444) observes: 'By [the close of the 17th century], arithmetic as we ordinarily speak of it, referring to the numbers for commercial and industrial purposes, was practically what it is today.' Though advances were still to come in the application of probability theory to financial valuation problems, as far as basic commercial arithmetic was concerned, D. Smith's view is essentially correct. Hence, based on this brief overview of the mathematical and historical background up to the close of the 17th century, what remains is to examine the practices that were contained in the commercial arithmetics up to that time. Aside from instruction on the basic operations of arithmetic and practical extensions such as the rule of three, commercial arithmetics were also filled with practical commercial problems. The solutions suggested to these problems are, arguably, the most important source of information on the accepted methods of calculation used by practitioners of the time.

Simple Interest in Partnerships

Methods of calculating interest are examined in the *Treviso* in three problems involving the returns from partnership (Swetz 1987, pp.138-9). No other attention is given to any situations involving interest payments. The first of these problems is an elementary application of the rule of three:

> Three merchants have invested their money in a partnership ... Piero put in 112 ducats, Polo 200 ducats and Zuanne 142 ducats. At the end of a certain period they found that they had gained 563 ducats. Required is to know how much falls to each man so that no one shall be cheated.

The solution to the problem is uninteresting from a mathematical viewpoint.[7] However, the problem is of interest in illustrating the general framework for practical partnership problems. In addition, the author is careful to implicitly observe that if all partners have funds invested for the same length of time, the solution to the problem is independent of the endpoint of the partnership.

The second of the *Treviso* problems is more complicated in that the partners are permitted to be involved in the partnership for different time periods (Swetz 1987, p.143):

> Two merchants, Sebastino and Jacomo, have invested their money for gain in a partnership. Sebastino put in 350 ducats on the first day of January, 1472, and Jacomo 500 ducats, 14 grossi on the first day of July, 1472; and on the first day of January, 1474 they found that they had gained 622 ducats. Required is the share of each (man so that no one shall be cheated).

The proposed solution to this problem follows as an extension of applying the rule of three given in the first problem. As such, this is also a simple interest method of solution. Observing that the stated solution does not admit the possibility of compound interest provides considerable insight into the methods of calculation used in mercantile practice during this period.

Considering the proposed solution in more detail requires knowing that 1 ducat = 24 grossi and 1 grossi = 32 pizoli. The solution to the problem proceeds by applying the rule of three which, in this case, involves expressing the two contributions in grossi, 8400 grossi for Sebastino and 12014 grossi for Jacomo with the addendum that 'since Sebastino has had his share in 6 months longer than Jacomo, we must multiply each share by the length of its time'. Multiplying by 24 months gives Sebastino's share as 201,600 and by 18 months gives Jacomo's share as 216,252. Taking the sum of these two shares (417,852) for a divisor and applying the 'rule of three' gives the solution of 300 ducats, 2 grossi, 8 pizoli and a remainder for Sebastino and 321 ducats, 21 grossi, 13 pizoli and a remainder for Jacomo.

The *Treviso* solution to the partnership problem does not involve the use of compound interest. Using semi-annual compounding, the inclusion of compound interest would involve solving:

$$850\frac{14}{24} + 622 = 350\left(1 + \frac{r}{2}\right)^4 + 500\frac{14}{24}\left(1 + \frac{r}{2}\right)^3$$

The solution of $r = 34.694\%$ requires the evaluation of a quartic equation. The associated shares would be 308.4 ducats (308 ducats, 9 grossi, 19 pizoli and remainder) for Sebastino and 313.6 ducats (313 ducats, 14 grossi, 12 pizoli and remainder) for Jacomo, a decidedly different result than the 'just' result proposed in the *Treviso*.[8]

While this failure to allow for compound interest was conventional in the early commercial arithmetics, there is evidence that accepted practice was not due to a general ignorance of the concept. For example, Italian manuscripts dating from the 14th and early 15th century Tuscany contained variations of the following problem: 'A man loaned 100 lire to another and after 3 years he gives him 150 lire for the principal and interest at annual compound interest. I ask you at what rate was the lira loaned per month?' (Franci and Rigatelli 1988). In other manuscripts, four year compound interest problems were proposed. The solutions proposed to these problems represent important contributions to the early development of algebra in Europe. Also of interest is Pegolotti (1936) which provides a 14th century Italian manuscript containing tables for the compound interest calculation $(1+r)^n$.

At least two possible factors can be identified for the failure to incorporate compound interest in valuing partnership returns, as reflected in 15th century and early 16th century commercial arithmetics. A first possible factor is simplicity of calculation. Even though the compound interest solution had been identified, the tables required for such calculations were not widely available, neither was the mathematical knowledge required for the merchant community to understand compound interest. While a reckoning master could be consulted on the 'just' solution to complicated problems, those involved in the day-to-day implementation of commercial arithmetic were primarily clerks and merchants. The *Treviso* algorithm, while inexact, only required applying the rule of three, a result that was at the heart of early commercial arithmetic.

Simplicity of calculation and general lack of mathematical knowledge imply that compound interest concepts were not typically incorporated into the business decisions of the time. Another possible factor supporting the *Treviso* solution was the usury restrictions imposed by canon law. While partnerships could be used to disguise the payment of simple interest, the explicit recognition of a 'profit on profit' payment could bring the sanctions of canon law upon those requiring the receipt of such a payment. For flagrant violation, these sanctions could include excommunication and even banishment. If such payments were made, and there is some anecdotal evidence that payment of compound interest was a regular business practice at the time of the *Treviso*, such payments were made in silence.

Strong evidence that the use of compound interest was common in the commercial practice of the 15th century, at least in the important financial centres such as Lyons, can be found in Chuquet's *Triparty*. On the subject of compound interest, the *Triparty* makes explicit reference to the incongruity between the theoretically correct mathematical calculation and recommended commercial practice for calculating shares in partnerships reflected in the basic commercial arithmetics. The manuscripts contained in the *Triparty* are actually three main sections concerned with algebraic theory, and three other parts containing problems, a geometry and a commercial arithmetic. The latter is generally similar in content to the *Treviso*, reflecting the similarity in the study of commercial arithmetic throughout Europe. However, unlike the *Treviso*, the handling of compound interest is recognized directly (pp.306-7):

> Three merchants formed a company, one of whom put in 10 ecus which remained there for the space of three years. The second put in 6 ecus which remained there for 7 years, and the third put in 8 ecus which remained there for four years. At the end of a period, 20 livres of profit was found. One asks how much comes to each, considering the money and the time that each has used it.

The answer proceeds with the usual application of the rule of three as in the *Treviso*. After presenting this method and the solution Chuquet states (p.307):

> And the calculation is done, according to the style and opinion of some. And in order for such reckoning to be of value, it is necessary to presuppose that the principal or the capital alone has made a profit, and not the profit (itself). And inasmuch as it is not thus, for the profit and the profit on the profit made in merchandise can earn profit and profit on profit in proportion to the principal, from day to day, from month to month and from year to year, whereby a larger profit may ensue. Thus such calculations are null, and I believe that among merchants no such companies are formed.

Though the compound interest solution is not provided, Chuquet definitely holds that calculation of compound interest is the regular practice in calculating the returns from partnerships of unequal duration.

The upshot of this discussion is that it is difficult to tell from an examination of the text of the basic commercial arithmetics such as the *Treviso* whether the use of compound interest in determining shares in partnerships was a widespread commercial practice. It is possible to rationalize the *Triparty* manuscript evidence indicating common usage of compound interest by arguing that social and commercial convention at the time did not permit acknowledging that compound interest calculations were used. In turn, the usury restriction was sufficiently binding that it would have been unwise to incorporate payment of compound interest, profit on profit, into the curriculum used to educate merchant apprentices. In addition, the mathematical concepts involved would have required a level of instruction substantively more advanced than that required to motivate fundamental concepts such as the 'rule of three'. On balance, it seems most likely that the practice of using compound interest to calculate partnership shares for investment periods of unequal duration was conventional and that the practice was not revealed in written sources order to avoid the usury sanctions.

Compound Interest

Problems involving compound interest were known well before the period under consideration. Mathematical tracts often used compound interest problems to motivate the solution of algebraic equations. For example, Franci and Rigatelli (1988, p.20) quote a 1395 Italian manuscript that poses the following problem:

> A man loaned 100 lire to another and after three years he gives him 150 for the principal and interest at annual compound interest. I ask you at what rate was the lira loaned per month?

The stated equation used to solve for this problem is the cubic equation ($x^3 + 60x^2 + 1200x = 4000$). The 1395 manuscript recognizes that the rule provided for solving the problem is not a general algebraic solution to cubic equations, but is applicable to the general interest rate calculation problems being posed.[9]

The level of algebraic sophistication in 14th century Italian mathematics extended beyond the solution of specific cubic equations. Quartic equations and, in very limited cases, higher order algebraic equations were also presented and solved. For example, the same 1395 Italian manuscript poses a compound interest problem requiring the solution of a quartic equation:

> A man loaned 100 lire to another and after four years he gives him for the principal and interest 160 lire at annual compound interest. I ask you are what rate the lira was loaned per month?

The associated quartic equation ($x^4 + 80x^3 + 2400x^2 + 32000x = 96000$) is again stated and solved with the recognition that the solution procedure to these types of compound interest problems is not algebraically general. This type of quartic problem reflects the ability of Italian mathematics of the time to solve algebraically for the interest rate in practical compound interest problems.

The use of compound interest problems to motivate algebraic solutions becomes increasingly common in mathematical manuscripts of the 15th and 16th centuries. However, as in the stated solution in the 1395 manuscript, compound interest problems were often used to illustrate the methods of algebra, not to facilitate commercial applications where the price-given-interest-rate calculation would typically involve substantially less mathematical skill. Widespread practical use of this type of compound interest calculation does require the availability of compound interest tables. The presence of such tables in printed commercial arithmetics or 'ready reckoners' is an excellent reflection of the extent of commercial use of compound interest valuations. Such tables do not begin to appear in print until about the mid 16th century, with detailed tables appearing only in the late 16th century.

As for the content of commercial arithmetic texts, until the later part of the 16th century compound interest problems are typically treated as mathematical, as opposed to commercial, problems. Basic commercial arithmetics, such as the *Treviso*, do not mention the subject. The treatment of compound interest was restricted to the more sophisticated mathematical texts, where a commercial arithmetic was included as one part of a text also dealing with the theoretical aspects of mathematics. Many of the mathematical texts that contain a section dealing with commercial arithmetic, such as the *Triparty*, either do include compound interest problems or include compound interest problems in a section other than that devoted to commercial arithmetic. Other texts, including Pacioli's *Summa*, provide a description of compound interest within the commercial arithmetic, without much elaboration.

Yet, it is fair to say that, at the time of the *Treviso*, the calculation of compound interest was recognized and understood. There were certainly reckoning masters with the skill required to do the required commercial calculations, if such calculations were needed, though compound interest rate problems were still mostly of interest to mathematicians. By the later part of the 16th century, compound interest calculations were much more widespread, as evidenced by the availability of tables needed to do compound interest calculations. From this time to the middle of the 17th century, progress in mathematics and commercial practice was substantial. Compound interest problems were no longer of much interest to mathematicians and commercial arithmetic gradually became the preserve of accountants and other specialized merchants.

Compound interest was still of mathematical interest in the 15th and early 16th century. Yet, the basic commercial arithmetics did not treat this subject. Given that these texts were designed for instruction in reckoning schools, this is not surprising. Merchants apprentices, struggling to understand the rule of three, could not be expected to have the mathematical skills to handle a concept that involved raising values to powers. Yet, many reckoning masters would have such conceptual ability. The implications of compound interest would be obvious to merchants, even if social restrictions prevented overt discussion. All this raises questions about the extent of compound interest calculations being used in practice,[10] with an important clue to solving this question being found in Chuquet's *Triparty*.

Seemingly, Chuquet's observations appears to call into question the validity of attributing the absence of compound interest problems in the early commercial arithmetics to the complexity of the solutions. Yet, the problem of determining a precise interest rate is more of a mathematical problem than a practical one. The mathematical motivation for compound interest problems is associated with solving for the rate of interest, given the starting and ending values of the investment. However, conventional practice was to state a rate of interest, and from this the ending or starting value for an investment could be readily calculated. Customary fixed interest rates were quoted regularly, one instance being the triple contract which was often referred to as a five-percent contract (Homer and Sylla 1991, p.75). Available interest bearing securities, such as annuities, mortgages, the *census* and the Venetian *prestiti*, typically offered annual coupon payments reducing the need to deal with compounding.

Where solutions to compound interest problems were required, reckoning masters such as Chuquet had the ability to make such calculations. For example, Chuquet poses the following problem:

> A merchant has lent to another a sum of money at the interest of 10%, and the interest earned like the principal at the end of every year. It happened that at the end of three years, the debtor is found to owe, as much in interest as in principal, the sum of 100 livres ... determine how much had been lent to him in the first year.

Chuquet's algebraic solution to this problem provides an answer that correctly incorporates the use of compound interest. Chuquet poses at least seven compound interest problems. Yet, none of these problems appears in the commercial arithmetic. Rather, these problems appear in a general section dealing with mathematical problems. Significantly, Chuquet continues the received practice of treating compound interest as being of mathematical, as opposed to commercial, importance.

More precisely, the *Triparty* has three main parts, dealing with arithmetic, calculation of roots and algebra. In addition to the main body of the *Triparty*, three supplementary sections are provided: a section with applied problems; a geometry; and, the commercial arithmetic (Flegg et al. 1985, p.197):

> Collections of mathematical problems, ranging from straight-forward calculations in fancy dress to purely logical brainteasers, have a long history, and played a prominent role in the transmission of mathematical culture throughout the Middle Ages ... In Chuquet's manuscript, the prime purpose of the Problems is ... to illustrate the applications of his *Triparty*, and in particular of the rule of first terms.

Though it is acknowledged that compound interest was common in commercial practice, and Chuquet has the ability to make the requisite calculations, the concept is still not included in the commercial arithmetic. This situation changes during the 16th century.

In a detailed examination of fourteen French commercial arithmetics written during the 16th century, Davis (1960, pp.22-4) finds ten of the fourteen dealing with problems of simple and compound interest, with two of the ten dealing with simple interest only. One of the ten arithmetics is *Larismethique* (1520, Lyons) by Etienne de la Roche, a probable student of Chuquet who also plagiarized liberally from the *Triparty*. La Roche's arithmetic makes a clear distinction between simple and compound interest:

> To merit [interest] is to make one's money earn or work in merchandise or otherwise at so much per livre or per cent at the finish of a year or of a month or of some other period. Simple merit [interest] occurs when the principal alone earns at the finish of the period. Merit at the finish of term [... compound interest] occurs when the principal earns at the end of the term, and then the gain and principal both earn ...

The following compound interest problem is then posed:

> A man lends another 100 livres for the space of two years and six months, to merit at the finish of term at the rate of twenty per cent. The question is what does it all amount to at the end of the term?

The solution la Roche offers does not proceed beyond methods provided by Chuquet.

In keeping with the still prevalent social restrictions on usury, four of the ten commercial arithmetics examining simple and compound interest contained criticism of the payment of interest but still discussed the subject because of the prevalence of the practice. One of these arithmetics was the amended 1561 French translation of the commercial arithmetic by important Dutch mathematician Gemma Frisius (1508-1555). Written originally in Latin around 1536, the Gemma arithmetic exhibited at least 59 editions in the 16th century and a number of further editions in the 17th century (D. Smith 1958, v.1, p.341). The amended French text states:

> Howsoever much this name of usury myst be execrable among Christians, nevertheless because necessity constrains many to this usage, I will speak a little of its computation.

Passing reference is made to compound interest as 'Judaic'.

The connection between compound interest and Jewish business practice is made in at least two other arithmetics. Jacques Chauvet in *Les Institutions de l'Arithmetique* (1578, Paris) describes compound interest as 'abominable' and says the practice is used only by Jews. However, this text is somewhat elementary. The more detailed work of Milles Denorry, *L'Arithmetique de Milles Denorry* (1574, Paris) refers to compound interest as 'Judaic usury' and observe that the practice was 'vituperable for Christians, thus punishable, and permitted only to Jews'. However, de Norry goes on to observe that compound interest had become 'so common that even the greatest were mixed up in it'. De Norry gives a full treatment to the calculation of compound interest (Davis 1960, p.24).

Of the four arithmetics examined by Davis that do not treat interest problems, two are the earliest considered, 1512 and 1515 respectively, and were printed in Paris, not Lyons which was the primary financial centre at that time. The other two date from the mid-century and were, again, printed in Paris. One of the authors, Pierre Forcadel, offers another commercial arithmetic that does treat compound interest problems. Of the two treating only simple interest, one is from Poitiers (1552) and the other is a 1578 French translation of Nicolas Tartaglia. As Tartaglia treats compound interest in other works, if Davis is correct the omission is one of text selection rather than lack of recognition by the primary author. Perhaps the most interesting commercial arithmetic examined by Davis (1960) is a 1515 arithmetic by a French monk and a Spanish monk printed in Lyons. This arithmetic deals with problems involving loans with late repayment and loans without interest. This arithmetic also contains an uncritical explanation of both simple and compound interest.

The Development of Commercial Arithmetic

As with much of early financial economics, it is difficult to trace the origins of specific valuation methods. One reason for this was a lack of attention given to these developments by the scholars of the time. This attitude gradually changed during the 16th century as commercial activities gained social importance. Within the university community, these activities were still largely considered within the realm of merchants and, with certain exceptions such as Petrus Ramus in Germany and Rudolf Snellus in Holland (van Berkel 1988), did not warrant the close attention of true scholars. In addition, certain valuation techniques were considered proprietary by the algorist or merchant firm involved and considerable effort was made to protect trade secrets. Where scholarly contributions were involved, the practice of plagiarism makes it difficult to correctly identify the originators of important developments.

As financial markets and instruments developed, so did the types of problems examined in the commercial arithmetics. Mathematical texts continued to examine the calculation of compound interest rates. For example, Tartaglia's *General Trattato* (1556) contained the following problem involving the interest rate applicable to a fixed term annuity:[11]

> A merchant gave a university 2814 ducats on the understanding that he was to be paid back 618 ducats a year for nine years, at the end of which the 2814

ducats should be considered paid. What compound interest did the merchant earn on his money?

As such problems were not often encountered in commercial practice, solving for the interest rate as the root of an algebraic equation was still primarily of interest to mathematicians. However, variation in commercial interest rates and the widening number of fixed income and other securities requiring valuation were creating a commercial demand for calculations involving compound interest. In turn, this demand was reflected in the emergence of tables for use in compound interest rate calculations.

One of the French arithmetics examined by Davis (1960) is the significant *L'Arithmetique* (1558, 1566) by Jean Trenchant printed in Lyons. One chapter of this text is concerned with simple and compound interest. The chapter features detailed calculations of annuity payments, as well as future and present values, for a given rate of interest. More importantly, Trenchant's publisher included in *L'Arithmetique* tables for annual compounding that are given for both future value, $(1+r)^T$, and the future value of an annuity, $[(1+r)^T - 1]/r$, evaluated at $r = 4\%$ for $T \in \{1,2,3,4,5,6\}$. Another table for $r=10\%$ for periods less than a year (in complete months) is also given (Lewin 1970). These tables are convincing evidence of the extent to which compound interest calculations were in general use in financial transactions.

Trenchant considers a range of problems involving compound interest (Lewin 1970). One problem involves a comparison of the relative value of a loan at 4% per quarter and an annuity of 5% per quarter, both covering 41 quarters. The loan is shown to be marginally superior. Another problem involves a merchant buying goods in exchange for an agreement to make payments totalling £1548, paid at the rate of £100 per year (15 yearly payments of £100 with a final payment of £48 in the 16th year). The problem is to find the present value of the payment stream (discounting at 17% per annum). Significantly, Trenchant determines the present value solution of £536 by individually discounting each of the cash flows and does not take advantage of the simplification provided by the closed form solution for the value of a term annuity. This 'brute force' method of calculation is used despite Trenchant's recognition, elsewhere, of the future value of an annuity reflected in the inclusion of the relevant Table.

Trenchant recognizes that compounding more frequently is desirable to the creditor, 4% per quarter is better than 16% per year. A problem is provided where the rent on a farm is £500 for three years and the present value of the payment stream is determined as £1105 using quarterly compounding:

$$\frac{500}{(1.04)^4} + \frac{500}{(1.04)^8} + \frac{500}{(1.04)^{12}} = 1105$$

Perhaps the most interesting problem encountered in the chapter is: 'If for 6000 one receives 7000 at the end of three years, how much would 100 increase by in one year?' Trenchant is able to provide an algebraic manipulation, involving the taking of a cube root, that solves the problem. However, the problem is only given in isolation. No attempt is made to generalize the solution procedure.

Trenchant's chapter on commercial arithmetic is far from being the most sophisticated of the 16th century. Much closer to this standard is the work of Simon Stevin (1548-1620) who is an early example of an important university mathematician drawn to solving practical financial valuation problems, complementing the work of the commercial algorists.[12] Simon Stevin was a Flemish mathematician working in Holland, credited with definitively introducing the decimal fraction into European mathematics (D. Smith 1958, v.1, p.343).[13] In one of the chapters in *La Practique d'Arithmetique* (1585), Stevin goes significantly beyond Jean Trenchant in providing tables for both present value $(1+r)^{-T}$ and present value of annuities $[(1+r)^T - 1)/(r(1+r)^T]$ for $T \in \{1,2,3,\ldots,30\}$ and $r \in \{1\%,2\%,\ldots,16\%\}$. Stevin also provides tables for the same time periods applicable to rates of interest for years' purchase of 15-22 years, that were of importance in the pricing of freehold properties.

Stevin does not provide future value of annuity tables but, instead, demonstrates the relationship between present value and future value, for both single cash flows and annuities. A table is provided for the future value of an annuity, $[(1 + r)^T - 1]/r$, associated with 15 years' purchase (6⅔%).[14] Stevin then proceeds to show how the present value and present value of annuity tables can be used to construct the future value of the annuity using the relation: $[(1 + r)^T - 1]/r = \{[(1+r)^T - 1]/[r(1 + r)^T]\} / \{(1 + r)^{-T}\}$. Stevin also demonstrates the method of using the tables for doing yield to maturity calculations. One example problem of this technique involves a term annuity problem: 'Someone owes £1500 p.a. to be paid over the next 22 years, and he pays his creditor £15,300 in lieu; what rate of interest does this represent?' Using the tables Stevin correctly identifies the solution as just more than 8%.

One interesting convention that Stevin uses is the practice of not considering less than annual compounding, providing a number of arguments against the practice. Where fractions of a year are involved, simple rather than compound interest is to be used. This convention is carried forward to valuing interest for problems where the term to maturity is greater than one year, but still involve a fraction of a year. For example, consider the solution posed to the problem:

> One wishes to know how much £800 capital, with its compound interest at a rate corresponding to 15 years' purchase, would amount to in 16½ years.

The solution given is:

$$[800\ (1 + r)^{16}] \left(1 + \frac{r}{2}\right) = £\ 2{,}321 \quad \textit{where } r = 6\tfrac{2}{3}\%$$

In this solution, compounding is applied for 16 years and then simple interest for the last half year.

Stevin's method of handling interest for periods of less than one year, where the full term exceeds one year, is not consistent with modern convention.[15] However, instead of being a reflection on the soundness of Stevin's technique, this divergence serves to illustrate the importance of convention in determining the specific methods that are used to make interest calculations. Stevin provided a number of sound arguments in defence of his treatment of interest calculations for fractions of a year. For example, Stevin observed that compound interest is intended to benefit the creditor. Yet, where fractions of a year are involved, compound interest will provide a smaller return than simple interest, which acts to the detriment of the creditor. Consequently, interest for fractions of a year ought not to be compounded.

Another argument provided by Stevin for using simple interest was that, because the prevailing convention was to calculate interest over a one year term or less using simple interest, then consistency required similar treatment whenever interest for fractions of a year are involved. Despite the soundness of Stevin's argument, the convention suggested is not reflected in modern practice. One possible explanation for this is the social aversion to usury. Stevin's convention would result in the highest possible interest payment by debtors. There is also the prevalence of legal maximum interest rates that were common in 16th and 17th century legislation permitting the payment of interest, for example, the 1545 Act in England that legalized interest payments and provided a legal maximum rate of 10%. Stevin's convention could, arguably, result in a technical breach of the maximum on loans that carried a stated interest rate equal to the maximum allowable.

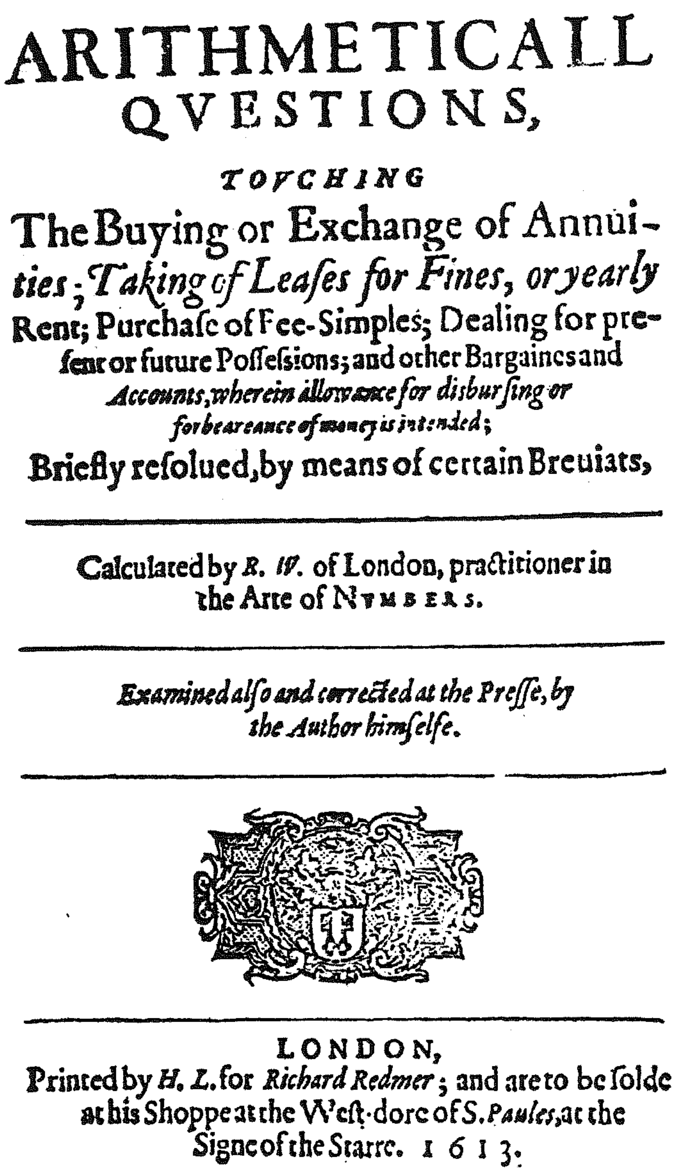
The Forgotten Work of Richard Witt

By the turn of the 17th century, the important financial centres of Europe had available commercial arithmetics, as well as academic arithmetics containing chapters on commercial arithmetic, written in the vernacular and dealing in considerable detail with the subject of compound interest. A number of sources contained tables needed to simplify compound interest calculations for practitioners. Further developments in the area of solving interest valuation problems involved broadening and deepening the subject matter, as well as increasing the distribution of advanced knowledge to lesser financial centres. The 17th century witnessed the emergence of texts written in the vernacular by commercial algorists that were technically advanced and dedicated solely to commercial interest rate calculations.

One important text that reflects the broadening and deepening of interest rate analysis was written by English commercial algorist Richard Witt, *Arithmeticall Questions, touching the Buying and Exchange of Annuities* ... (1613, London) (Lewin 1970) (Figures 5.1 and 5.2). The history of this book is something of an enigma. By standards of early 17th century commercial arithmetic, the contents of the book are sophisticated. The book was considered to be of enough significance to appear in a second edition in 1634. This second edition appeared after the death of Witt and was produced by Thomas Fisher who made a number of additions to the original text. However, Fisher observes in his introduction to the book the 'the Book is almost forgot and out of use'. One element of the enigma surrounding this book is why it failed to have much staying power or later notoriety.

One possible reason for the lapse into obscurity is that the book became a victim of what Thomas Fisher described in his introduction as the 'change of times and customs'. It is also possible that the book was too much for most practitioners, who could get what was required from more accessible sources, such as the tables of Thomas Clay or, somewhat later, from the ready reckoner of William Leybourn. In turn, little is known of Richard Witt other than his description on the title page as a 'practitioner in the Arte of Numbers'. He was almost

Figure 5.1 Title page from Witt (1613)



ARITHMETICALL
QVESTIONS,

TOVCHING

The Buying or Exchange of Annui-
ties; *Taking of Leaſes for Fines, or yearly*
Rent; Purchaſe of Fee-Simples; Dealing for pre-
ſent or future Poſſeſsions; and other Bargaines and
Accounts, wherein allowance for disburſing or
forbeareance of money is intended;
Briefly reſolued, by means of certain Breuiats,

Calculated by *R. W.* of London, practitioner in
the Arte of NVMBERS.

Examined alſo and corrected at the Preſſe, by
the Author himſelfe.

LONDON,
Printed by *H. L.* for *Richard Redmer*; and are to be ſolde
at his Shoppe at the Weſt-dore of S. *Paules*, at the
Signe of the Starre. 1613.

Source: This image was adapted from Lewin (1970).

Figure 5.2 The first question, with the worked solution, from Witt (1613)

The firſt Queſtion, being an example of the firſt Direction.

If 1li.*be put forth at Intereſt after* 10.per Cent.per Annum, *Intereſt, and Intereſt vpon Intereſt, for* 30. *yeares; Vnto how much will it amount by the end of that time?*

BEcauſe the time in this Queſtion is 30.yeares, looke in the Breuiat next before ſet downe, for the thirtieth number; which you ſhall finde to be 174494022. from this cut off 7.figures, beginning to tell from your right hand towards your left, & then it will ſtand thus: 17|4494022. The 17.that ſtandeth on your left hand is 17li.

Now Multiplie 4494022.(the 7.figures cut off) by 20.and the product will be. 89880440. from this alſo cut off 7.figures, and then it wil ſtand thus 8|9880440. The 8.on your left hand is 8 ſh.

Now Multiplie 9880440.(the figures laſt cut off) by 12.and the product will be 118565280. from this alſo cut off 7.figures, and it will ſtand thus 11|8565280. The 11.on your left hand is 11 d.

So haue you found, that if 1li.be put forth at Intereſt after 10.*per Cent. per Ann.* Intereſt, and Intereſt vpon Intereſt for 30.yeares, it will amount by the 30. yeares end, vnto 17 l.8 ſh.11 d.

The Worker

Facit	l	17	4494022
	ſh	8	988044
	d	11	856528

Source: This image was adapted from Lewin (1970).

certainly alive in 1613 when the first edition was printed and, according the Thomas Fisher's introduction to the second edition, had died by 1634. Witt, apparently, lacked any desire for self-promotion, a trait that probably extended to promotion of *Arithmeticall Questions*.[16]

Witt's book has two significant features: a sequence of detailed interest tables, that Witt refers to as 'breviats'; and 124 problems that are solved using the tables.[17] The detail and sophistication exhibited in the interest tables is impressive. *Arithmeticall Questions* starts with a discussion of the relationship between the various tables for future and present value, both for single cash flows and annuities. A future value table listing the factors for $(1 + r)^T$ for $r = 10\%$ and $T \in \{1,2,\ldots,30\}$ is provided. Following this table is a demonstration of how to use to the values in the table to construct the associated factors for present value of single cash flows, present value of an annuity and future value of an annuity. The 10% rate of interest is important because this was the prevalent rate in England at that time for financial transactions other than those involving land. After demonstrating the calculations, complete present value, present value of annuity, and future value of annuity tables for $r = 10\%$ and $T \in \{1,2,\ldots,30\}$ are provided.

The $r = 10\%$ case was important because of the practical importance of calculations involving this rate. Having demonstrated how to calculate various factors from the future value tables, Witt also provides future value tables for a range of less practically important interest rates, $r \in \{9\%, 8\%, 7\%, 6\%, 5\%\}$ for $T \in \{1,2,\ldots,30\}$. Lewin (1970, p.124) observes that: 'The other functions (for present value, present value of annuity and future value of annuity) are not quoted, however, a lack of which was evidently felt by at least one reader, because the British Museum has a copy of the book in which there has been inserted contemporary manuscript tables that give the missing functions at length'. As the calculations involved in land valuation conventionally were done using 16 years' purchase, Witt provides a complete set of present and future value tables for 6¼%.

Witt's work proceeds substantially further than just providing more detailed tables than those given by Stevin. In particular, Witt goes beyond Stevin in considering less than annual compounding frequencies. Future value tables for $(1 + r)^{T/2}$ and $(1 + r)^{T/4}$ are given for odd values of T, for the practical interest rates of $r = 10\%$ and $r = 6¼\%$. Other tables relevant for less than annual compounding frequencies are also given.

The content of certain problems is another feature of *Arithmeticall Questions* that goes beyond Stevin. For example, Witt gives the following problem (Q.99):

> A man hath a Lease of certaine grounds for 8 years yet to come: for which he payeth £130 per Ann. Rent, viz. £65 per halfe yeare: which grounds are worth £300 per Ann. viz. £150 per halfe yeare. If this man shall surrender-in his Lease; what ready mony shall he pay with it to his Land-lord for a new Lease of 21 years, not altering the Rent of £130 per Ann. reckoning such int. as men have when they buy Land for 20 years' purchase, and receive the Rent halfe yearly?

The solution requires recognizing that 20 years' purchase translates to 5% interest which is 2½% 'halfe yearly', an interest rate for which Witt provides a table. The answer of £1085 1s 6d now follows because the landlord will have to forego an annuity of £85 per half year for 13 years.

Two other problems provide useful examples of the level of sophistication in Witt's problems. Q.70 poses the following valuation problem:

> One oweth £900 to be paid all at the end of 2 yeares: he agreeth with his Creditor to pay it in 5 yeares, viz. every yeare a like summe. They demaund what each of these 5 payments shall be, reckoning 10 per Cent. per Ann. int. and int. upon int.

The solution requires the future value of £900 to be discounted to the present value and, then, the annuity payment to be determined by solving a present value of annuity problem. In this fashion, Witt determines the correct annuity payment of £196 4s 3d.

Perhaps the most interesting solution given to the various problems posed by Witt is associated with Q.103:

> A oweth to B £1200 to be paid in 6 yeares, in 12 equall payments, viz. at the end of each halfe yeare £100. They agree to cleare this debt in 3 yeares, in 6 equall payments, viz. at the end of each halfe yeare, one payment. The Question is, what each payment ought to be, reckoning interest after the rate of 10 per Cent per Ann. and int. upon int.

A conventional solution to this problem can be determined by equating the discounted value of the annuity stream of £100 for 12 half-year periods with the discounted value of £C for 6 half-year periods and solving for C. The exact solution requires recognizing Witt's practice of using $(1 + r)^{T/2}$ instead of the modern convention of $(1 + r/2)^T$ to discount the T period cash flow.

More precisely, the solution can be determined by solving:

$$\frac{100}{(1 + r)^{1/2}} + \frac{100}{(1 + r)} + ... + \frac{100}{(1 + r)^6}$$

$$= \frac{C}{(1 + r)^{1/2}} + \frac{C}{(1 + r)} ... + \frac{C}{(1 + r)^3}$$

$$= 100 \left[1 + (1 + r)^{1/2}\right] \left[\frac{1}{r} - \frac{1}{r (1 + r)^6}\right]$$

$$= C \left[1 + (1 + r)^{1/2}\right] \left[\frac{1}{r} - \frac{1}{r (1 + r)^3}\right]$$

Solving this for r = 0.10 gives the solution stated by Witt of £175.13145 or £175 2s 7d. Yet, Witt is able to show that this solution can be obtained as:

$$100 + \frac{100}{(1 + r)^3} = £175\ 2s\ 7d$$

Lewin (1970, p.126) describes the method Witt uses to arrive at this solution as 'extremely elegant'.

From a careful consideration of the tables and problems, Lewin (1970, p.128) concludes:

> it is clear that by 1613, the techniques of compound interest were no longer still in their infancy. It was accepted that compound interest should be allowed in ordinary business and legal transactions, and the methods of carrying out the arithmetic were clearly understood. The differences between simple and compound interest were fully appreciated, as well as the difference between, for example, a rate of 10% per annum and a rate of 2½% per quarter.

One additional interesting feature of *Arithmeticall Questions* is the absence of any problems that involve solving for a yield; not even for integer value interest rate problems, let alone the more complicated variants that involve interpolating between factors listed in appropriate tables. The solution of such problems does have a long history within the more mathematical stream of commercial arithmetic. However, Lewin (1970, p.130) is probably correct in stating: 'it may be that there was little call for this in practice'.

Malynes on Compound Interest

Malynes's *Lex Mercatoria* (1622) is not a commercial arithmetic. However, the *Lex Mercatoria* does reflect the impact that commercial arithmetic had on the acceptance and sophistication of interest calculations of that period. Chapter 14, 'Of the true Calculation of Moneys delivered at Interest', examines the 'absurditie' of the legal interpretations for the then-prevailing maximum annual interest rate of 10%. The legal requirement involved the payment of simple interest only. On a one year loan contract at 10%, £100 would earn only £10. Malynes recognizes the ability to legally earn more than 10% in one year by, instead of lending using a one year contract, a sequence of three month contracts could be used, with the principal incremented by the amount of interest paid as each contract takes effect. An examination of the discussion in the *Lex Mercatoria* is useful as it reflects the general level of merchant knowledge of compound interest, whereas the *Arithmeticall Questions* better reflects the level of specialist knowledge.

Malynes (1622, p.346) then considers the same strategy applied to longer term loans:

> In like manner is it for moneys delivered out for a longer Time; as for example, one delivered out £100 for four years, for which at the four years end he can receive but £140; but if he had delivered out £100 for one year, he may at years end receive £10 for Interest, and continue the £100 pounds for the second year by a new agreement, and then receive another 10 pound, and so on for the third and fourth year. Now whereas by reason of his several agreements according to time, he hath altered the property of Interest money, and received £10 the first year, he may put out again this £10 as his own for another year, and so have Interest thereof twenty shillings, whereby he receiveth £11 the second year, which being put out the third and fourth year, will yield him after the same manner accordingly: so that he shall have above £146 being thus delivered out, the body of his sum still remaining whole ...

While the exact answer of £146 and 8 shillings is not provided, the concept of compound interest is clearly understood as is the ineffectiveness of a maximum legal interest rate based on simple interest. The ability to make payment at 10% compound interest is a matter of correctly structuring the loan contracts. Similar problems are observed for term annuity payments.

Despite the recognition and appreciation of compound interest in *Lex Mercatoria*, the contrast between Malynes and Witt is striking. Though the first edition of *Arithmeticall Questions* predates the first edition of *Lex Mercatoria* by almost a decade, the former is almost modern by comparison. Witt, the algorist, demonstrates a high level of sophistication in making compound interest calculations while Malynes, the merchant, is still quite concerned about the usury restrictions. Four full chapters and parts of other chapters in *Lex Mercatoria* are dedicated to various aspects of usury, with only one brief chapter dedicated to 'the true calculation of moneys at interest'. Complicated interest problems, of the sort discussed by Witt, are not considered. It would seem that, when involved interest calculations were required in the early 17th century, the task would almost certainly be undertaken by a specialist.

Compound Interest Calculations in the 18th Century

Despite considerable facility with compound interest within the restricted community of specialists directly involved in making interest calculations, by the end of the 18th century Halley (1693) found it necessary to include a present value table for $r = 6\%$ and $T \in \{1,2,3,\ldots,100\}$ in the text of the 'Estimate...'. In effect, the process of refining and expanding the available tables needed for compound interest calculations continued into the 18th century. This process was expanded to include tables that could be used to calculate the present values of life annuities. In addition to work on life annuities and related problems by de Moivre, Simpson, Price and others, the traditional problem of calculating present value and future value interest tables continued. In 1726, John Smart published his comprehensive *Tables of Interest, Discount and Annuities*. These tables, taken to nine significant figures, are credited (Pearson 1978) with being the primary source for tabular interest calculations in de Moivre, Simpson and others.

Present value tables are only useful in solving certain types of problems. In particular, it is difficult to use tables to calculate yields from prices. In addition, in Halley's time there was also the problem of lack of tables with the precision needed to do specific problems, for example, calculations involving an unconventional interest rate. In the

'Estimate' (pp.602-3), Halley explicitly recognizes the ability to handle many present value calculations using logarithms:

> Now the present value of Money payable after a term of years, at any given rate of Interest, either may be had from Tables already computed; or almost as compendiously, by the Table of Logarithms: For the Arithmetical Complement of the Logarithm of Unity and its yearly Interest (that is, of 1.06 for Six *per Cent.*) being 9,974694.

This suggested use of logarithms to derive solutions to present value problems was an extension of Halley's mathematical interest in the computation of logarithms. In 1698, Halley was to publish a paper in *Philosophical Transactions*: 'A most compendious and facile Method for Constructing the Logarithms, without regard to the Hyperbola, with a speedy method for finding the Number from the Logarithm given'. Pearson (1978, p.87) credits this paper for also containing 'the first proof of what we now call the exponential theorem', the result that e^x is the limit, as n tends to infinity, of $[1 + (1/n)]^{nx}$.

In 1704, Halley was appointed Savilian Professor of Geometry at Oxford University. Sometime during his tenure at Oxford Halley wrote 'Of Compound Interest', a comparatively scarce work that was published in *Sherwin's Mathematical Tables* (1761). In this work, Halley provides numerous worked examples of the use of logarithms to solve present value problems involving single cash flows and annuity streams. Halley states his rationale for the article as: 'A principal use of logarithms is to solve all cases of compound interest which are not, without great difficulty, attainable by the rules of common arithmetic'. That Halley would advocate and illustrate the use of logarithms in the context of practical fixed income calculations reflects a number of possible observations. In particular, by the early 18th century mathematical knowledge of practitioners had expanded to include logarithms. The practical use of logarithms was enabled by the widespread availability of logarithmic tables. Also, practitioners were sometimes confronted with more complicated calculations where logarithms would be useful.

Halley initially starts with a single cash flow problem. More precisely, a single cash flow of £15 17s 6d is paid in order to receive £31 18s 10½d in twelve years time which translates into 6% compound interest. Halley illustrates the method of solving for one of the values, when the other three are given. For example, remembering that Halley follows the convention of his time and uses 'r' to represent $(1 + i)$:

emergence of the monied commercial and rentier classes seeking investment outlets.

Appendix: Life Annuity and Other Tables from Price (1772)

The availability and sophistication of tables used in present value calculations is an important measure of the degree of development that financial calculations had achieved at a given point in time. As a benchmark for the state of development of financial calculations at the end of the period under consideration, a sampling of the tables included in Price (1772) is provided (Figures 5.3 and 5.4). It is not surprising, given Price's concern with life annuity valuation problems, that various life tables are included. In total, the collection of tables is an expanded collection of the tables provided by de Moivre, *A Treatise of Annuities on Lives*. De Moivre provides: four different life tables, by Halley, Kersseboom, Deparcieux, and Smart/Simpson; a table for the 'Sums of Logarithms' from 10 to 900; five present value of annuity tables for $r \in \{3\%, 3.5\%, 4\%, 5\%, 6\%\}$ and $T \in \{1,2,3,\ldots,100\}$; and, five corresponding tables for 'The present Value of an Annuity of one Pound, to continue so long as a Life of a given Age is in being', for ages from 1 year to 86 years with rates of interest $r \in \{3\%, 3.5\%, 4\%, 5\%, 6\%\}$.[18]

Price includes fourteen tables in his appendix. In addition, there are a number of other tables scattered here and there throughout the text. Of the fourteen tables, the first two are for present value of a single cash flow and present value of an annuity. Five different interest rates are given: 3, 3½, 4, 5 and 6 percent. The frequency is annual from 1 to 100, with the annuity table also giving the value of a perpetuity. Tables III-V contain different life tables, one by Halley and two others, one for Northampton and one for Norwich. The first page of Table VI is reproduced here as Figure 5.3. This table gives the present value of a life annuity 'according to Mr. De Moivre's hypothesis'. Table VII gives the present value of a joint life annuity, again according to Mr De Moivre's hypothesis.

Tables VIII-IX are of considerable historical interest. Table VIII is a life table, the first page of which is reproduced here as Figure 5.4. This table explicitly recognizes the contribution of Mr Simpson for deducing the life table 'from observations on the bills of mortality in London for 10 years, from 1728 to 1737'. It is now believed that this work was primarily the product of John Smart, though it appears that Simpson was given the credit at the time that Price was writing. Tables

Figure 5.3 Table of life annuity values from Price (1772)

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TABLE VI. (a).

Shewing the preſent Values of an annuity of 1*l.* on a ſingle Life, according to Mr. *De Moivre*'s hypotheſis; and, therefore, nearly, according to the probabilities of life at BRESLAW, NORWICH, and NORTHAMPTON. See p. 2, and 206.

Age.	3 per Ct.	3½ per Ct.	4 per Ct.	4½ per Ct.	5 per Ct.	6 per Ct.
8	19,736	18,160	16,791	15,595	14,544	12,790
9	19,868	18,269	16,882	15,672	14,607	12,839
10	19,868	18,269	16,882	15,672	14,607	12,839
11	19,736	18,160	16,791	15,595	14,544	12,790
12	19,604	18,049	16,698	15,517	14,480	12,741
13	19,469	17,937	16,604	15,437	14,412	12,691
14	19,331	17,823	16,508	15,356	14,342	12,639
15	19,192	17,707	16,410	15,273	14,271	12,586
16	19,050	17,588	16,311	15,189	14,197	12,532
17	18,905	17,467	16,209	15,102	14,123	12,476
18	18,759	17,344	16,105	15,015	14,047	12,419
19	18,610	17,220	15,999	14,923	13,970	12,361
20	18,458	17,093	15,891	14,831	13,891	12,301
21	18,305	16,963	15,781	14,737	13,810	12,239
22	18,148	16,830	15,669	14,641	13,727	12,177
23	17,990	16,696	15,554	14,543	13,642	12,112
24	17,827	16,559	15,437	14,442	13,555	12,045
25	17,664	16,419	15,318	14,340	13,466	11,978
26	17,497	16,277	15,197	14,235	13,375	11,908
27	17,327	16,133	15,073	14,128	13,282	11,837
28	17,154	15,985	14,946	14,018	13,186	11,763
29	16,979	15,835	14,816	13,905	13,088	11,688
30	16,800	15,682	14,684	13,791	12,988	11,610
31	16,620	15,526	14,549	13,673	12,855	11,530
32	16,436	15,367	14,411	13,553	12,780	11,449
33	16,248	15,204	14,270	13,430	12,673	11,365

(a) This Table is the ſame with Mr. *De Moivre's* Table of the values of ſingle lives, publiſhed in his *Treatiſe on Life Annuities*, and carried as far as the age of 79 to 3 places of decimals by Mr. *Dodſon* in his *Mathematical Repoſitory*, vol. 2. p. 169.

Source: Adapted from Price (1772).

Figure 5.4 One of the life tables contained in Price (1772)

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TABLE VIII.

Shewing the Probability of the Duration of life in LONDON, deduced by Mr. *Simpson* from observations on the bills of mortality in LONDON for 10 years, from 1728 to 1737.

Ages.	Persons living.	Decr. of Life.	Ages	Persons living.	Decr. of Life.	Ages.	Persons living	Decr. of Life.
0	1000	320	27	321	6	54	135	6
1	680	130	28	315	7	55	129	6
2	547	51	29	308	7	56	123	6
3	496	27	30	301	7	57	117	5
4	469	17	31	294	7	58	112	5
5	452	12	32	287	7	59	107	5
6	440	10	33	280	7	60	102	5
7	430	8	34	273	7	61	97	5
8	422	7	35	266	7	62	92	5
9	415	5	36	259	7	63	87	5
10	410	5	37	252	7	64	82	5
11	405	5	38	245	8	65	7[illegible]	5
12	400	5	39	237	8	66	7[illegible]	5
13	395	5	40	229	7	67	6[illegible]	5
14	390	5	41	222	8	68	62	4
15	385	5	42	214	8	69	58	4
16	380	5	43	206	7	70	54	4
17	375	5	44	199	7	71	50	4
18	370	5	45	192	7	72	46	4
19	365	5	46	185	7	73	42	3
20	360	5	47	178	7	74	39	3
21	355	5	48	171	6	75	36	3
22	350	5	49	165	6	76	33	3
23	345	6	50	159	6	77	30	3
24	339	6	51	153	6	78	27	2
25	333	6	52	147	6	79	25	
26	327	6	53	141	6			

Note: This table demonstrates the common perception that Simpson was the source of the life table that probably originated with Smart.

Source: This image was adapted from Price (1772).

IX, X and XI develop the information in Table VIII: Table IX giving the 'Expectations of Life in London', Tables X and XI giving the single and joint life annuity prices derived from the supposed Simpson life table.

Tables XII-XIV are concerned with providing information from an updated life table for London, 'formed from the Bills, for 10 years, from 1759 to 1768'. Table XII gives the life table, Table XIII derives the 'true Probabilities of Life in London 'till the age of 19', and Table XIV extends the results to all ages.

Notes

1. To be most effective, modern interest calculations require the cash flows to be determinate. This permits important measures such as the yield to maturity to be estimated. The yield to maturity is an important market measure for comparing bond values. As the internal rate of return, the yield to maturity is also used in capital budgeting decisions. However, the yield to maturity embeds an assumption that the coupon cash flows can be reinvested at the offered yield to maturity. This means that the yield to maturity is only indicative, it is a 'promised yield to maturity' and not the actual return which will be earned. Many of the fixed income securities traded in the sixteenth to eighteenth centuries were not readily adaptable to the convention that the cash flows were determinate. Redeemable annuities would often be redeemed when state finances permitted, meaning that the term to maturity was not certain. Even perpetual annuities would be redeemed under certain circumstances. Life annuities offered an even more indeterminate cash flow pattern. For these type of securities, interest measures such as the yield to maturity were not as useful. Hence, both years' purchase and the more modern measures of relative value lack exactness. (Years' purchase would have difficulty valuing a zero coupon bond but this type of security was not common in early financial markets.)

2. One reason that US discount rates differ from true yields is the convention that the year has 360 days. This results in the unusual (365/360) adjustment when pricing a one year security.

3. Simple interest does not involve compounding, that is, the payment of interest on interest. For example, assume that \$100 is invested at 4% a year for five years, using simple interest, with all payments made at maturity. In five years time, the investor would receive \$120, the \$100 return of principal together with \$4 per year for five years.

4. To use the perpetual solution to solve for the term annuity certain lasting until T, observe that the price of a term annuity can be determined by subtracting the current price of an annuity starting in year T from the price of a perpetual. The price of the perpetual is $1/r$ and the price of the perpetual starting at T (first payment at $T + 1$) is $1/r$ discounted back at $1/(1 + r)^T$.

5. In modern applications, the series solution to $\log[1 + x]$, where log is the natural log, is used to justify the use of x for $\log[1 + x]$, when x is sufficiently small, as in the case of an interest rate. For example, this approximation could be used in approximating the domestic currency return on a foreign asset ($R_\$$): $(1 + R_\$) = (1 + R_£)(1 + e)$, where e

is the exchange rate return and $R_£$ is the return denominated in foreign currency terms. Using the log approximation, it follows that $R_\$$ can be set approximately equal to $R_£ + e$. From this, a less complicated closed form for the variance of the domestic currency return on a foreign asset can be derived.

6. N. Mercator is not the same individual as Gerardus Mercator (1512-1594), the Flemish cartographer and mathematician who devised, in 1568, the Mercator projection in mapping. Mercator projections were an important advance in navigation as maps of this type allowed a straight-line on the map to plot a course which could be followed without changing compass direction. The series solution to $\log[1 + x]$ provided by N. Mercator was presented prior to the introduction of the calculus.

7. The stated solution is: for Piero, 138 ducats, 21 grossi, 11 pizoli and remainder; for Polo, 248 ducats, 0 grossi, 13 pizoli and remainder; and, for Zuanne, 176 ducats, 2 grossi, 7 pizoli and remainder. The *Treviso* proceeds to check the solution, so that 'no one has been cheated', by adding together the shares to verify that the total is 563 grossi.

8. The third problem is a more complicated variation of the second: 'Three men, Tomasso, Domenego, and Nicolo, entered into partnership. Tomasso put in 760 ducats on the first day of January, 1472, and on the first day of April took out 200 ducats. Domenego put in 616 ducats on the first day of February, 1472, and on the first day of June took out 96 ducats. Nicolo put in 892 ducats on the first day of February, 1472, and on the first day of March took out 252 ducats. And on the first day of January, 1475, they found that they had gained 3168 ducats, 13 grossi and 1/2. Required is the share of each, so that one shall be cheated.' The solution procedure is an extension of the rule-of-three procedure used to solve problem 2. However, due to crediting Nicolo with three months full investment instead of only one month, 'the solution stated does not satisfy the given conditions of the problem' (Swetz 1987, p.147). Ignoring the remainders, the solution is given for Tomasso as, 1052 ducats 11 grossi and 8 pizoli, for Domenego, 942 ducats 3 grossi and 21 pizoli, and for Nicolo, 1173 ducats 22 grossi and 17 pizoli.

9. To see how this cubic equation solves the problem posed requires some further discussion. Franci and Rigatelli observe: 'The most general formulation of the problem is the following: Calculate at what rate the *lira* was loaned per month knowing the capital is *A lire*, and after three years *B lire* are given back. Further interest must be added to the capital at the end of each year.

Let *x denari* be the rate of one *lira* per month. If we remember that one *lira* is equal to 240 *denari*, we obtain the equation: $x^3 + 60\,x^2 + 1200\,x = 8000\,((B/A) - 1)$.' This approach can be compared with the expansion of the modern form of the pricing problem originally posed:

$$100 = \frac{150}{(1 + \frac{i}{12})^3} \rightarrow \frac{150}{100} - 1 = 3\,\frac{i}{12} + 3\,\frac{i}{12}^2 + \frac{i}{12}^3$$

In the coinage of the time, a *denari* was the same as a grossi, this equivalence originating from the more formal *denari de grossi*. The coinage used further required 20 soldi = 1 lira and 12 grossi = 1 soldo. From this, the relation of 1 *lira* with 240 *denari* is explained. The cubic equation stated for arriving the appropriate solution now follows by solving the modern cubic in terms of *denari*, which requires grossing up by 240. But, in order to obtain a monthly rate which requires division of the annualized interest rate by twelve, the equation is only multiplied through by $(20)^3 = 8000$.

10. There are various practical instances where such calculations would be required. For example, a merchant may want to compare the promised return on a one year investment with the return which would be earned on a six month investment followed by a reinvestment of principal plus interest in another six month investment.

11. Guillaume Gosselin (1578, Paris) is a French translation, with additional notes by Gosselin, of the Nicolas Tartaglia arithmetic, the *General Trattato*. It is unclear why this work is examined by Davis (1960) and identified as dealing in simple interest problems only. Smith (1970, p.278) makes reference to the *General Trattato* as: 'Indeed, there is no other treatise that gives as much information concerning the arithmetic of the sixteenth century, either as to theory or application. The life of the people, the customs of merchants, the struggles to improve arithmetic, are all set forth by Tartaglia in an extended by interesting fashion.' As Tartaglia did consider compound interest in the *General Trattato*, the Davis (1960) claim about the omission of compound interest is either a mistake or is due to Gosselin's editing of the original text in translation and is not to Tartaglia's lack of knowledge in this area.

12. Stevin is sometimes referred to using the Latin form, Stevinus. D. Smith (1970, p.386) refers to *La Practique d'Arithmetique* as 'an attempt at a practical textbook, but too scholarly for its purposes.' The interest tables which are included in Stevin (1585) were produced and published three years earlier.

13. There is usually some room for debate over the introduction of notational advances, such as the decimal fraction. De Morgan (1847) claims: 'I now hold it next to certain that the same convenience which has always dictated the decimal form for tables of compound interest was the origin of the decimal fractions themselves.' In effect, Stevin's 1582 production of compound interest tables can be credited as the original contribution for introducing decimal fractions into European mathematics. Though there is evidence that decimal fractions were used by Chinese and Arabic mathematics some time earlier, up to Stevin's time the practice was unknown in Europe.

14. The calculation of this interest rate has to do with the convention used to specify interest rates. Because years purchase is price divided by annuity payment ($YP = P/A$) and the annuity payment can be calculated by the price times the stated interest rate ($A = rP$), it follows that years purchase can be approximated as $YP = 1/r$. Hence, normalizing the price to 100, years purchase times the interest rate (taken as a whole number) will give 100. There is no connection being made between the term to maturity of the loan and the interest rate, as in the PVIFA. Using modern compound interest calculations, to warrant a price of 100, an annual payment of $6\frac{2}{3}$ would require a term of well over 40 years to justify an interest rate of $6\frac{2}{3}\%$.

15. Instead of calculating $800(1 + r)^{16.5}$, modern convention would probably do the calculation as $800\,[1 + (r/2)]^{33}$.

16. It is possible to make some conjectures about Witt's possible background. The surname 'Witt' is not a common English name, this surname being more common in the Low Countries. Witt's presence in England around the end of the beginning of the 17th century is consistent with the hypothesis that he was part of the mass emigration from Antwerp and environs associated with the various conflicts which affected that area at the end of the 16th century. This wave of skilled emigration affected many individuals

involved in the early history of financial economics. For example, the family of Joseph de la Vega emigrated from Antwerp first to Germany and, later, settled in Amsterdam. Gerard de Malynes was also, most likely, part of this emigration. The sophistication of Witt's analysis would have required advanced training. Such training would have been difficult to obtain in England. Such training would have been available in Antwerp during its heyday as the commercial centre of Europe. If Witt had obtained such training in England and then proceeded to develop an active practice as 'a practitioner of numbers', it is likely that some paper trail would have been left. No such trail has yet been unearthed.

17. The breviats attracted the attention of de Morgan (1847, p.575) as being an early contribution to the use of decimal fractions which predates Napier by four years.

18. Recalling that the 'Treatise...' is appended to the third edition of the *Doctrine of Chances*, the table for the Sum of Logarithms is included in the Appendix to the whole text. This table is used to solve a specific problem from the *Doctrine* and is not of direct relevance to the 'Treatise...'.